Non-critical bosonic string corrections to the black hole entropy

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Abstract

We calculate the quantum corrections to the entropy of a very large black hole, coming from the theory of a D-dimensional, non-critical bosonic string. We show that, for $D > 2$, as a result of modular invariance the entropy is ultraviolet finite, although it diverges in the infrared (while for $D = 2$ the entropy contains both ultraviolet and infrared divergences). The issue of modular invariance in field theory, in connection with black-hole entropy, is also investigated.

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The issue of black-hole entropy —together with its puzzling consequences— is under permanent discussion nowadays. It is commonly recognized by now that the main arguments and definitions were already given in the classical works [1]. However, twenty years have elapsed since then and many points remain still mysterious.

Focussing on our specific interest in the subject, it is well known [2] that the black-hole entropy calculated in quantum field theory near the horizon diverges in the ultraviolet. A few months ago, some proposals have been made about the fact that string theory (considered as a model for a theory of quantum gravity) might be indeed relevant for the description of a very large black hole [3, 4]. In particular, recently the quantum correction to the entropy of a large black hole in critical string theory has been calculated in [5]. It has been shown there that, as a result of modular invariance (which is associated with the gauge symmetry in string theory), the string corrections to the entropy are ultraviolet finite —unlike what happens in the field theoretical case. However, infrared divergences appear, that are most probably connected with the Hagedorn temperature [6] (for a recent review on string theory at non-zero temperature see [7]).

In the present letter we extend the approach of refs. [3]–[5] and calculate the quantum correction to the entropy in a D-dimensional, non-critical bosonic string. As in the case of the critical bosonic string, for any $D > 2$ the correction is ultraviolet finite but it contains infrared divergences. Then, a comparison is done with the corresponding quantum corrections to the entropy that are obtained in field theory —this entropy being also presented in modular invariant form.

We start with the derivation of the formula for the entropy in the theory of a non-critical bosonic string in D dimensions. Presumably, this will correspond to the expression for the one-loop correction to the entropy of a very large black hole in a D-dimensional bosonic string medium.

The expression of the one-loop free energy for a non-critical, D-dimensional, bosonic string has been given in a modular invariant form in ref. [9]. It is the following

$$
\begin{aligned}
F(\beta) \;=\; & -\int_{\mathcal{F}} \frac{d^2\tau}{8\pi^2\tau_2^2} \left|\eta\left(e^{2\pi i\tau}\right)\right|^{-2(D-2)} \left(4\pi^2\tau_2\right)^{-(D/2-1)} \\
& \times \sum_{n,m=-\infty}^{\infty} \exp\left[-\frac{\beta^2}{8\pi\tau_2}(m^2 + n^2|\tau|^2 - 2\tau_1 nm)\right],
\end{aligned}
\tag{1}
$$

where β is the inverse temperature, $\eta(x) \equiv x^{1/24}\prod_{n=1}^{\infty}(1 - x^n)$, D is the physical dimension

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($D = 26$ corresponds to the critical bosonic string), and $\mathcal{F}$ is the fundamental domain:

$$|\tau| > 1, \quad -1/2 < \text{Re } \tau < 1/2, \quad \text{Im } \tau > 0. \tag{2}$$

Notice that a very similar expression would have been obtained, had we considered the non-critical bosonic string with the dynamical Weyl mode [10]. However, in the explicit formula an additional piece in the integrand appears (see for example [11]), which depends on the radius of the compactified Weyl mode and leads to some problems of interpretation (as that of a continuous string spectrum). Another remark has to do with the fundamental region $\mathcal{F}$. If one would have performed the calculation of the free energy in the field theory of string modes, the integration region in (1) would be completely different from $\mathcal{F}$ (see nevertheless what follows). However, it is well known [12] that by doing so one erroneously overcounts the string result and a transformation of this new region to the fundamental one $\mathcal{F}$ must therefore be performed in any case.

Now we can turn to the calculation of the entropy —more precisely, of the corrections to the entropy coming from our non-critical string. The semiclassical expression for the black hole entropy that is widely accepted nowadays is [1]

$$S_{BH} = \frac{A}{4\hbar G}, \tag{3}$$

where G is the gravitational constant and A is the area of the event horizon. In order to obtain the entropy in the bosonic non-critical string theory under consideration, we follow ref. [5] where the previous proposals of refs. [4, 3] have been further developed. We first calculate the entropy density as

$$S(\beta) = \beta^2 \frac{\partial F(\beta)}{\partial \beta}, \tag{4}$$

where $F(\beta)$ is given by (1). Supposing now that, very near to the horizon, the external observer can be approximated by a Rindler observer with a position-dependent temperature, we substitute $\beta = 2\pi z$ to obtain the local entropy density. Again, following ref. [5] we may calculate the total entropy as

$$\begin{aligned} S &= \int_0^\infty S(z)\, A\, dz = \pi A \int_0^\infty dz\, (2\pi z)^3 \int_{\mathcal{F}} \frac{d^2\tau}{8\pi^2 \tau_2^2} \left| \eta\left(e^{2\pi i \tau}\right) \right|^{-2(D-2)} \\ &\quad (4\pi^2 \tau_2)^{-D/2} \sum_{n,m=-\infty}^{\infty} (m^2 + n^2|\tau|^2 - 2\tau_1 nm) \exp\left[-\frac{\pi z^2}{2\tau_2}(m^2 + n^2|\tau|^2 - 2\tau_1 nm) \right]. \end{aligned} \tag{5}$$

Expression (5) is modular invariant for arbitrary D and, as a result (and owing to the fact that Im $\tau = 0$ is excluded from the integration region in (5)), the entropy is ultraviolet

finite at any value of $D > 2$ (for the case of the critical bosonic string, where $D = 26$, this fact was already noticed in ref. [5]). Again, as for the bosonic critical string, expression (5) includes the infrared divergence produced by the tachyon. Hence, entropy is divergent in the infrared limit. Very likely, this issue is closely connected with the Hagedorn phase transition [6] and, numerically, with its associated Hagedorn temperature [6, 7], but we do not know yet how to establish this relation explicitly. However, we observe that, concerning the infrared divergence, after properly taking care of the zero mode ($n = m = 0$ in (5)), a high-β expansion can be performed in (5) by using Jacobi's theta function identity and expanding the η function (these techniques are described in [13]). As is easily seen from this perturbative expansion, the expression for the free energy becomes convergent for $\beta > \beta_c = 2\pi\sqrt{(D-2)/3}$. This value of β_c is the one associated with the Hagedorn temperature.

Expression (5) is rather complicated, and it is quite difficult to analyze in detail, in order to extract the full information that it contains. In general, only a few qualitative features can be obtained from it, as the property of ultraviolet finiteness mentioned already. Nevertheless, in some special situations an analytic treatment is possible. For example, let us consider the case when $D = 2$ (the two-dimensional string). Then, the same calculation as in (1) can be performed, what yields a very nice result, as follows [8, 9]

$$F^{D=2}(\beta) = \frac{1}{6\sqrt{2}\,\beta}\left(\frac{2\pi\sqrt{2}}{\beta} + \frac{\beta}{2\pi\sqrt{2}}\right). \tag{6}$$

The dual symmetry of the usual bosonic string [12] is clearly seen from (6).

Calculating the entropy for $D = 2$, one gets

$$S^{D=2} = \frac{1}{3}A\ln\frac{l}{\varepsilon}, \tag{7}$$

where l is an infrared cut-off and ε an ultraviolet one. In its structure, this result is very similar to the one obtained in ref. [4], where in fact the same expression for the entropy was gotten for dilatonic gravity (apart from the factor A in (7)). It must not seem strange that the calculated correction to the entropy is divergent in both the infrared and the ultraviolet regions, since at $D = 2$ the string is effectively reduced to a $D = 2$ field theory, with the known consequences.

We will now compare the above result (5) for the black-hole entropy with the corresponding result coming from D-dimensional bosonic field theory. In this case the calculation of entropy may be performed as shown before, and leads to the following result [5]

$$S(m^2) = A\int_0^\infty dz\,(2\pi z)^3 \int_0^\infty \frac{ds}{s^2}(2\pi s)^{-D/2}\sum_{\tau=1}^\infty \tau^2 \exp\left(-\frac{m^2 s}{2} - \frac{2\pi^2\tau^2 z^2}{s}\right), \tag{8}$$

where m is the boson mass. As was pointed out in [4, 3], there is an ultraviolet divergence in (8), since the s integral grows without bound near $s = 0$ when z is small. For the massless case and $D = 2$ we can easily obtain from (8) an exact expression which has the same form as (7), except for an additional factor $1/2$. In fact, for $m^2 = 0$,

$$S(m^2 = 0) = (2\pi)^{3-D/2}A \int_0^\infty dz\, z^3 \int_0^\infty ds\, s^{-2-D/2} \sum_{\tau=1}^\infty \tau^2 \exp\left(-\frac{2\pi^2\tau^2 z^2}{s}\right), \tag{9}$$

and performing the change of variables $s \to t = 2\pi^2\tau^2 z^2/s$ we get immediately

$$S(m^2 = 0) = 2^{2-D}\pi^{1-3D/2}\zeta(D)\Gamma(D/2+1)A \int_0^\infty dz\, z^{1-D}, \tag{10}$$

in particular,

$$S^{D=2}(m^2 = 0) = \frac{1}{6}A \ln \frac{l}{\varepsilon}. \tag{11}$$

In the general case an explicit expression under the form of a quickly convergent series of McDonald's functions is obtained

$$S(m^2) = (2\pi)^{3-D/2}A \int_0^\infty dz\, z^3 \sum_{\tau=1}^\infty 2\tau^2 \left(\frac{m}{2\pi\tau z}\right)^{D/2+1} K_{D/2+1}(2\pi m\tau z), \tag{12}$$

which can then be approximated and safely truncated after the first term [13]:

$$K_{D/2+1}(2\pi m\tau z) \sim \begin{cases} \frac{1}{2}\Gamma(D/2+1)(\pi m\tau z)^{-D/2-1}, & z \text{ small}, \\ \frac{1}{2}(m\tau z)^{-1/2}\exp(-2\pi m\tau z), & z \text{ not small}. \end{cases} \tag{13}$$

Substituting (13) into (12) we see that, in this massive case and for the upper limit of the entropy integral we obtain a finite result

$$\int^\infty \sim (2\pi)^{-(D+1)/2}\zeta(2)\Gamma((5-D)/2)Am^{D-2}, \tag{14}$$

in particular, for $D = 2$,

$$\int^\infty \sim \frac{\pi A}{24\sqrt{2}}, \tag{15}$$

while in the lower integration limit, the same behaviour as for the massless case appears

$$\int_0 \sim -\frac{A}{6}\ln\varepsilon. \tag{16}$$

It is interesting to observe, on the other hand, that quantum corrections to the entropy in field theory can be actually presented under the form of a modular invariant quantity also. (Of course, the origin of SL(2,Z) invariance in field theory has this formal character [14],

albeit not a fundamental one as in string theory, where it is associated with gauge symmetry itself). Starting from the expression for the free energy of a D-dimensional massive boson

$$F(\beta) = -\int_0^\infty \frac{ds}{s}(2\pi s)^{-D/2} \sum_{\tau=1}^\infty \exp\left(-\frac{m^2 s}{2} - \frac{\tau^2 \beta^2}{2s}\right), \tag{17}$$

let us formally introduce in (17) a unity as follows

$$1 = \sum_{k=1}^{N_0} \delta_{k1} = \sum_{k=1}^{N_0} \int_{-1/2}^{1/2} dt \, \exp[2\pi i t(k-1)], \tag{18}$$

where $N_0 > 1$ is an arbitrary natural number. Then, we can write,

$$F(\beta) = -\sum_{k=1}^{N_0} \int_0^\infty \frac{ds}{s}(2\pi s)^{-D/2} \int_{-1/2}^{1/2} dt \, \exp[2\pi i t(k-1)] \sum_{\tau=1}^\infty \exp\left(-\frac{m^2 s}{2} - \frac{\tau^2 \beta^2}{2s}\right), \tag{19}$$

and by choosing now $t = \mathrm{Re}\,\tau$, $s = \mathrm{Im}\,\tau$, $\tau = \tau_1 + i\tau_2$, it is easy to see that (19) is invariant under the change $\tau \to \tau + 1$, i.e. it is invariant under the group of discrete translations U. But, as is known, U is the congruence (Borel subgroup) of the modular group SL(2,Z). Thus, using the techniques of ref. [14] for the field theory propagator, we can actually rewrite (19) in modular invariant form:

$$F(\beta) = -\sum_{k=1}^{N_0} \sum_{(c,d)=1} \int_{\mathcal{F}} d^2\tau \, \frac{\exp[2\pi i(k-1)\mathrm{Re}\,\gamma_{cd}\tau]}{\mathrm{Im}\,\gamma_{cd}\tau(2\pi \mathrm{Im}\,\gamma_{cd}\tau)^{D/2}} \sum_{\tau=1}^\infty \exp\left(-\frac{m^2}{2}\mathrm{Im}\,\gamma_{cd}\tau - \frac{\tau^2 \beta^2}{2\mathrm{Im}\,\gamma_{cd}\tau}\right), \tag{20}$$

where the second sum is extended over relative prime integers c, d, $\mathcal{F}$ is again the fundamental domain, and

$$\gamma_{cd} = \begin{pmatrix} * & * \\ c & d \end{pmatrix}, \tag{21}$$

with the $*$ denoting arbitrary elements —in the sense that any transformation of SL(2,Z) with the bottom row (c,d) can be used as a representative of the coset element.

From here one can easily obtain, just in the same way as in (8), the entropy corresponding to the bosonic field theory in a modular invariant form:

$$\begin{aligned} S &= A\int_0^\infty dz\,(2\pi z)^3 \sum_{k=1}^{N_0} \sum_{(c,d)=1} \int_{\mathcal{F}} d^2\tau \, \frac{\exp[2\pi i(k-1)\mathrm{Re}\,\gamma_{cd}\tau]}{(\mathrm{Im}\,\gamma_{cd}\tau)^2(2\pi \mathrm{Im}\,\gamma_{cd}\tau)^{D/2}} \\ &\quad \sum_{\tau=1}^\infty \tau^2 \exp\left[-\frac{m^2}{2}\mathrm{Im}\,\gamma_{cd}\tau - \frac{\tau^2(2\pi z)^2}{2\,\mathrm{Im}\,\gamma_{cd}\tau}\right]. \end{aligned} \tag{22}$$

In this manner we have been able to calculate the quantum correction to the entropy in the bosonic field theory as a modular invariant expression, what contradicts the claim against

this possibility made in ref. [5]. Although very similar in structure to (8), expression (22) appears to be free from the ultraviolet divergence. In fact, even if z can still be zero, the reason for the ultraviolet divergence in (8) was the growth of the second integral (over s there). This seems not to be the case now, since all points in the integration region $\mathcal{F}$ are at distance ≥ 1 from the origin of the τ-plane. As we have seen in detail before (eqs. (11) and (16)), tecnically the appearance of the singularity was drawn by a change of variables in the second integral that left the integration range unchanged and introduced negative powers of z in the first, rendering it divergent. Again, this cannot be done here without altering the $\mathcal{F}$ region. Summing up, we think that the entropy in its modular invariant form (22) does no exhibit the usual ultraviolet divergence. This should not mean that it is absolutely free from divergences, because a possible appearance of an new one, which could come from the sum over c, d (again for very small z) is not excluded without more carefull work. Indeed, expression (22) seems to be quite difficult to analyze in general, although the hope remains that it can yield workable results in some particular cases, by using specific zeta-function techniques (as the ones introduced in ref. [13]).

In conclusion, we have calculated here the quantum corrections to the entropy for the case of a non-critical bosonic string. Such entropy is modular invariant in any dimension D and it is ultraviolet finite for $D > 2$. We have also found a modular invariant expression for the entropy in bosonic field theory. Recently, a new representation (in terms of a Laurent series) for the string free energy has been introduced in ref. [15]. Quite remarkably, this representation gives a convenient way to perform the analytical continuation of the free energy beyond the Hagedorn temperature. However, modular invariance is not explicitly aparent in this description, owing mainly to the fact that it is not an integral representation. It would be, nevertheless, of interest to analyze the divergences of string entropy in that formalism [15], what we are planning to do in the near future.

Acknowledgments

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